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Journal of Sound and Vibration 276 (2004) 1135–1144
JOURNAL OF SOUND AND VIBRATION
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Letter to the Editor

Radial vibration characteristics of piezoelectric cylindrical transducers

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Received 1 April 2003; accepted 13 November 2003

1. Introduction

Piezoelectric transducers, based on piezoelectric phenomenon [1] which converts electric signals into mechanical vibrations and vice versa, are used as electromechanical sensors or actuators in various fields [2]. Most transducers use longitudinal vibration in the thickness direction of a plate or a disc. In rare cases of disc use, the piezoelectric torsional transducer uses torsional vibration with shear motion in the circumferential direction [3].

On the other hand, piezoelectric cylindrical transducers have been introduced in several forms. A transducer polarized in the axial direction undergoes axial motion under the electric drive in the radial thickness direction, and is used as an aligner or a translator, as for example in a scanning tunnelling microscope [2]. A transducer polarized in the circumferential direction undergoes radial vibrations resulting from circumferential expansion and compression [4].

This paper deals with the radial vibration of piezoelectric cylindrical transducers polarized in the radial direction. The behaviors of these transducers have been studied in some different point of view [5,6]. Transducers polarized in the circumferential direction [4] are different from these transducers, even though they both vibrate radially. The transducers considered in this paper were installed on a pipe, and axisymmetric waves in the pipe wall were generated or detected [7]. The purpose of this paper is to establish a formula for calculating the piezoelectric natural frequency of these transducers.

First of all, the differential equations of piezoelectric radial motion were derived in terms of radial displacement and electric potential. The characteristic equation of radial vibration was obtained by applying mechanical and electric boundary conditions. Theoretical calculations of the fundamental natural frequency were compared with the experimental observations for transducers of several sizes. The dependence of the piezoelectric natural frequency on the radius and thickness of the piezoelectric cylinder is discussed.

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doi:10.1016/j.jsv.2003.11.015

The equations in the analysis were simplified to include radial displacement and radial variables only, and the material properties were simplified to be isotropic ones. This simplification restricts the validity of the analysis to the materials whose anisotropic factors are close to 1, but it allows convenient prediction of the radial vibration characteristics according to the geometry of the piezoelectric cylinders. Similar simplification is found in other literature dealing with a piezoelectric sphere [8].

2. Theoretical analysis

Electromechanical relationships were determined for a piezoelectric disc vibrating in the thickness direction [9]. A similar scheme is introduced for modelling the piezoelectric cylindrical transducer, schematically shown in Fig. 1. The piezoelectric cylinder has uniform electrodes on the inner surface of radius R_i and on the outer surface of radius R_o. Radial vibrations in the cylinder can be described in terms of the axisymmetric radial displacement $u(r,t)$ and electric potential $\phi(r,t)$, both functions of the radial coordinate r and time t.

The components of radial stress σ_r and circumferential stress σ_θ in the piezoelectric cylinder are expressed as the stress components in an elastic cylinder [10] with the effect of the radial piezoelectricity [9] as follows:

$$\sigma_r = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2G\frac{\partial u}{\partial r} + e\frac{\partial \phi}{\partial r}, \tag{1}$$

$$\sigma_\theta = \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + 2G\frac{u}{r}. \tag{2}$$

Piezoelectric materials are anisotropic, but they can be approximated to elastically isotropic when the elastic anisotropy factor is close to 1. The radial component of electric displacement D_r, incorporating piezoelectric effect, is expressed as follows:

$$D_r = e\frac{\partial u}{\partial r} - \varepsilon\frac{\partial \phi}{\partial r}. \tag{3}$$

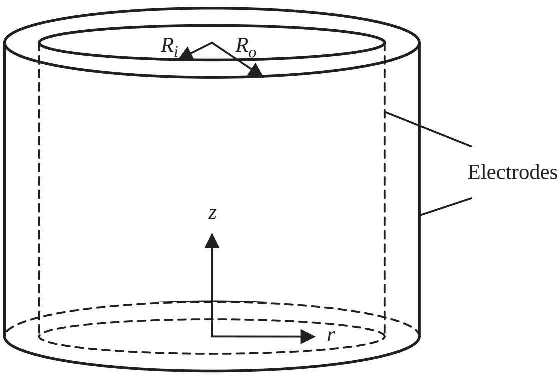

Fig. 1. Schematic diagram of a cylindrical transducer.

Here, λ and G are Lamé elastic constants, which are expressed in terms of Young's modulus E and Poisson ratio v as $\lambda = Ev/(1+v)(1-2v)$ and $G = E/2(1+v)$. In addition, $e\ (= d_{33}/s_{33}^{E})$ is the piezoelectric stress constant and ε the dielectric permittivity [9].

Inserting Eqs. (1)–(3) into the equations of motion and the electrostatic equation in cylindrical co-ordinates yields the following equations:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{E_L}\frac{e^2}{\varepsilon}\frac{u}{r^2} = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2}, \tag{4}$$

$$\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) = \frac{e}{\varepsilon}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \tag{5}$$

where $c_L\ (= [E_L/\rho]^{1/2})$ in Eq. (4) is the propagation speed of the longitudinal wave, ρ is the mass density, and $E_L = \lambda + 2G + e^2/\varepsilon$.

When the voltage applied to the electrodes is a harmonic function of time t with frequency ω, the displacement u and the electric potential ϕ are regarded as harmonic functions of time with the same frequency. Therefore, $u(r,t)$ and $\phi(r,t)$ can be expressed through the separation of variables as follows:

$$u(r,t) = \tilde{u}(r)\mathrm{e}^{\mathrm{i}\omega t}, \qquad \phi(r,t) = \tilde{\phi}(r)\mathrm{e}^{\mathrm{i}\omega t}. \tag{6a, b}$$

Substituting Eqs. (6a) and (6b) into Eqs. (4) and (5) provides the following governing equations:

$$r^2\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}r^2} + r\frac{\mathrm{d}\tilde{u}}{\mathrm{d}r} + (k^2r^2 - p^2)\tilde{u} = 0, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}r}\right) = \frac{e}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\tilde{u}}{\mathrm{d}r}\right), \tag{8}$$

where $k\ (= \omega/c_L)$ is the wave number, and p is a constant, defined as follows:

$$p^2 = 1 - \frac{1}{E_L}\frac{e^2}{\varepsilon}. \tag{9}$$

The solution of Eq. (7) has the following form:

$$\tilde{u}(r) = AJ_p(kr) + BJ_{-p}(kr). \tag{10}$$

After inserting Eq. (10) into Eq. (8), the solution of $\tilde{\phi}(r)$ is obtained as follows:

$$\tilde{\phi}(r) = \frac{e}{\varepsilon}[AJ_p(kr) + BJ_{-p}(kr)] + a\ln r + b. \tag{11}$$

The unknown constants A, B, a, and b are determined according to the boundary conditions.

As shown in Fig. 1, the piezoelectric cylinder has an inner radius of R_i and an outer radius R_o. The transducer is driven by an electric voltage $V_0\mathrm{e}^{\mathrm{i}\omega t}$ applied to its inner and outer surfaces. Boundary conditions are established as follows:

$$\tilde{\sigma}_r = 0 \quad \text{and} \quad \tilde{\phi} = 0 \quad \text{at } r = R_i, \tag{12a, b}$$

$$\tilde{\sigma}_r = 0 \quad \text{and} \quad \tilde{\phi} = V_0 \quad \text{at } r = R_o. \tag{12c, d}$$

Since the radial stress σ_r $(=\tilde{\sigma}(r)e^{i\omega t})$ has the formula as stated in Eq. (1), applying boundary conditions (12a)–(12d) to Eqs. (10) and (11) yields a set of equations with unknown constants A, B, a, and b.

Eliminating the constants a and b in the equations results in a set of two equations in a matrix form as follows:

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ V_0 \end{bmatrix}, \tag{13}$$

$$D_{11} = R_o f_1(kR_o) - R_i f_1(kR_i),$$

$$D_{12} = R_o f_2(kR_o) - R_i f_2(kR_i),$$

$$D_{21} = g_1(kR_o) - g_1(kR_i) - \frac{R_o}{e} f_1(kR_o) \ln \frac{R_o}{R_i},$$

$$D_{22} = g_2(kR_o) - g_2(kR_i) - \frac{R_o}{e} f_2(kR_o) \ln \frac{R_o}{R_i}$$

where

$$f_1(kr) = \hat{E}_L \frac{\mathrm{d}}{\mathrm{d}r}[J_p(kr)] + \lambda \frac{1}{r} J_p(kr), \tag{14a}$$

$$f_2(kr) = \hat{E}_L \frac{\mathrm{d}}{\mathrm{d}r}[J_{-p}(kr)] + \lambda \frac{1}{r} J_{-p}(kr), \tag{14b}$$

$$g_1(kr) = \frac{e}{\varepsilon} J_p(kr), \tag{14c}$$

$$g_2(kr) = \frac{e}{\varepsilon} J_{-p}(kr). \tag{14d}$$

The unknown constants are determined by obtaining constants A and B from Eq. (13) and inserting them into the original equations

$$A = -\frac{V_0}{\Delta}[R_o f_2(kR_o) - R_i f_2(kR_i)], \tag{15a}$$

$$B = \frac{V_0}{\Delta}[R_o f_1(kR_o) - R_i f_1(kR_i)], \tag{15b}$$

$$a = \frac{V_0}{\Delta}\frac{R_i R_o}{e}[f_1(kR_i) f_2(kR_o) - f_1(kR_o) f_2(kR_i)], \tag{15c}$$

$$\begin{aligned} b = \frac{V_0}{\Delta}\{ & [R_o f_2(kR_o) - R_i f_2(kR_i)] g_1(kR_i) \\ & - [R_o f_1(kR_o) - R_i f_1(kR_i)] g_2(kR_i) \\ & - \frac{R_i R_o \ln R_i}{e}[f_1(kR_i) f_2(kR_o) - f_1(kR_o) f_2(kR_i)]\}, \end{aligned} \tag{15d}$$

where Δ represents the determinant of the matrix in Eq. (13).

Resonance occurs when the determinant Δ is equal to 0:

$$\Delta \equiv \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = 0. \tag{16}$$

Eq. (16) is the characteristic equation representing the resonance of a piezoelectric cylindrical transducer driven in the radial direction. Meanwhile, the mode shapes of the radial vibration can be obtained by inserting Eqs. (15a) and (15b) into Eq. (10) and assuming a value of 1 for V_0/Δ for a relative displacement distribution.

3. Comparison with experiment

The results of the analysis described in the previous section can be verified by calculating the natural frequencies and comparing them with experimental observations. The unknown variable k in Eq. (16) can be calculated easily by using a root-finder function (FindRoot) available in Mathematica [11]. A successful search necessitates a good initial guess, which can be selected by the elastic natural frequency of a corresponding non-piezoelectric, i.e., elastic, cylinder. Once the wave number k is evaluated, the natural frequency f is obtained from the following relation:

$$f = \frac{kc_L}{2\pi}. \tag{17}$$

The piezoelectric material selected for the numerical calculation and experiment was PZT (EC-64), manufactured by EDO Co. The material properties are as summarized in Table 1, and they are similar to the values reported in other literature [12]. The properties converted in terms of the expressions in this paper are as in Table 2. Three transducers A, B, and C of different sizes were used in the research. Their outer radius R_o, inner radius R_i, and thickness T are seen in Table 3. The lengths of the transducers A, B, and C shown in Fig. 2 were 20, 15, and 12 mm, respectively, but these values were unnecessary in the calculations. The piezoelectric natural frequencies of the fundamental mode for these transducers shown in Table 3 were calculated from Eq. (16).

In order to verify the calculated values of the piezoelectric natural frequencies, measurements were carried out with the piezoelectric circular transducers as shown in Fig. 2. The resonance frequency of a transducer was measured using the Impedance Gain/Phase Analyzer (HP 4194A). The measured impedance displayed as a function of the frequency is shown in Fig. 3. The locations of local minimum impedance in the curve of Fig. 3 represent the piezoelectric natural frequencies. The measured and calculated piezoelectric natural frequencies are shown in Table 3.

As seen in Table 3 the calculated and measured values agree well with each other. Therefore, the analysis described in the previous section appeared to explain accurately the vibration characteristics of piezoelectric cylindrical transducers. Particularly, it could be of use in the design stage in determining the size of a transducer for a particular frequency.

Table 1
Material properties of a PZT (EDO EC-64)

Properties		Values
Mechanical	Mass density, ρ	7500 kg/m^3
	Elastic compliance, s_{33}^E	0.0150×10^{-9} m^2/N
	Elastic compliance, s_{44}^E	0.0388×10^{-9} m^2/N
Dielectric	Relative permittivity, $\varepsilon_{11}^S/\varepsilon_0$	692
Electromechanical	Charge constant, d_{33}	0.295×10^{-9} C/N

Table 2
Converted properties of a PZT (EDO EC-64)

Properties		Values
Mechanical	Young's modulus, E $(= 1/s_{33}^E)$	66.7 GPa
	Shear modulus G $(= 1/s_{44}^E)$	25.8 GPa
	Poisson ratio, ν $(= E/2G - 1)$	0.293
	Lamé constant, λ (Eq. (4a))	36.4 GPa
Dielectric	Permittivity of a free space, ε_0	8.854×10^{-12} C^2/N m^2
	Permittivity, ε	6.130×10^{-9} C^2/N m^2
Electromechanical	Piezoelectric stress constant, e $(= d_{33}/s_{33}^E)$	19.67 C/m^2

(A) (B) (C)

Fig. 2. Photograph of three transducers.

4. Discussions

By using the analysis described and verified in the previous sections, numerical calculations were carried out to determine mode shapes, the effect of the piezoelectricity on the natural

Table 3
Comparison of the calculated and measured natural frequencies of the fundamental mode for transducers of three sizes

Transducer	Size (mm)		Fundamental frequency (kHz)	
	Outer radius	Inner radius	Calculated	Measured
A	14.3	12.0	37.9	38.8
B	10.05	7.80	56.0	56.3
C	7.10	5.50	79.4	80.8

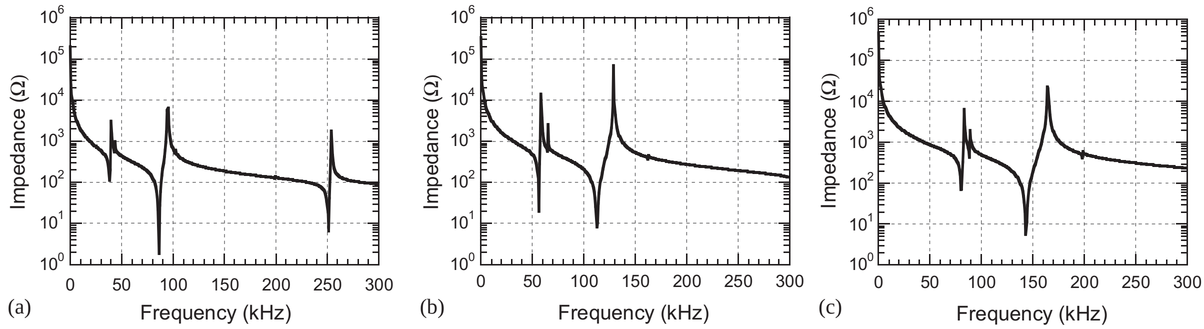

Fig. 3. Impedance curves of the piezoelectric transducers, as measured as a function of frequency; (a) transducer A, (b) transducer B, (c) transducer C.

frequency, and the dependence of the piezoelectric natural frequency on the radius and thickness of the cylinder.

4.1. Mode shapes

Mode shapes of the radial vibration can be obtained by inserting Eqs. (15a) and (15b) into Eq. (10) and assuming a value of 1 for V_0/Δ for a relative displacement distribution. The fundamental mode calculated for transducer A, as seen in Fig. 4, shows a similarity to a rigid-body mode of a plate.

4.2. Dependence of natural frequency on the cylinder radius

The piezoelectric natural frequency of the fundamental mode was calculated and displayed as a function of cylinder radius with the solid line in Fig. 5. In addition, the natural frequency of an elastic cylinder, represented by a dashed line in Fig. 5, is compared with the piezoelectric natural frequency, represented by a solid line. The thickness of the cylinder was fixed to 2 mm.

The two curves in Fig. 5 coincide, indicating that piezoelectricity does not affect the natural frequency of the fundamental mode. This phenomenon can be explained by observing that the natural frequency approaches zero as the radius increases and by considering that the fundamental mode corresponds to a rigid-body mode of a plate. The mode shape in Fig. 4

supports this speculation. In other words, the fundamental mode has motion without deformation in the thickness direction of a plate with infinite radius of curvature. Furthermore, its natural frequency increases as the radius of curvature decreases, with some gradual accompaniment of deformation.

4.3. Dependence of natural frequency on the cylinder thickness

In order to investigate the dependence of the piezoelectric natural frequency of the fundamental mode on the thickness of the cylindrical transducer, as shown in Fig. 6 the frequency was calculated as a function of the thickness for fixed values of the mean radius R_m ($= (R_i + R_o)/2$). As can be seen, the frequency varies little with the thickness. This trend can be explained by the observation that the fundamental mode corresponds to a rigid-body mode of no thickness deformation as mentioned in Sections 4.1 and 4.2.

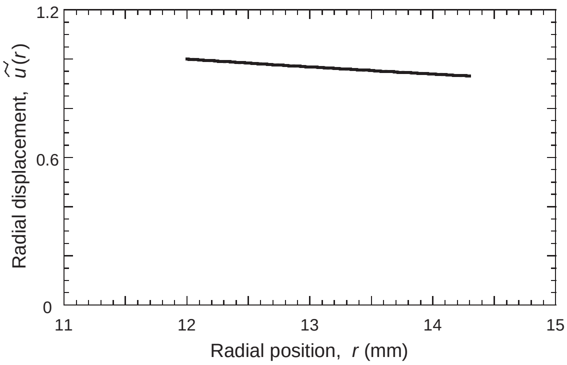

Fig. 4. Fundamental mode shape of the radial vibration of transducer A.

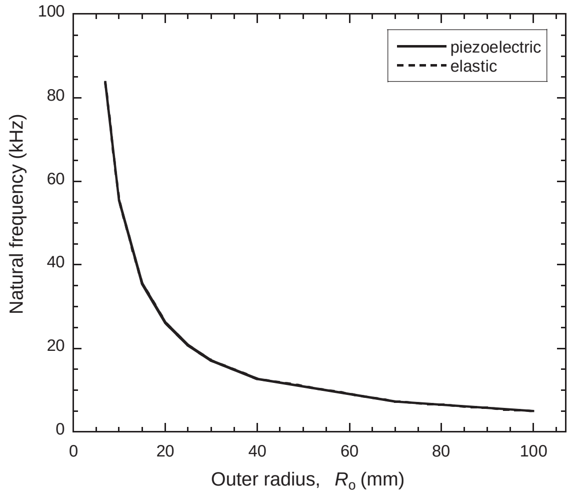

Fig. 5. Comparison of the piezoelectric and elastic natural frequencies of the fundamental mode displayed as a function of the outer radius of the cylinder with a thickness of 2 mm.

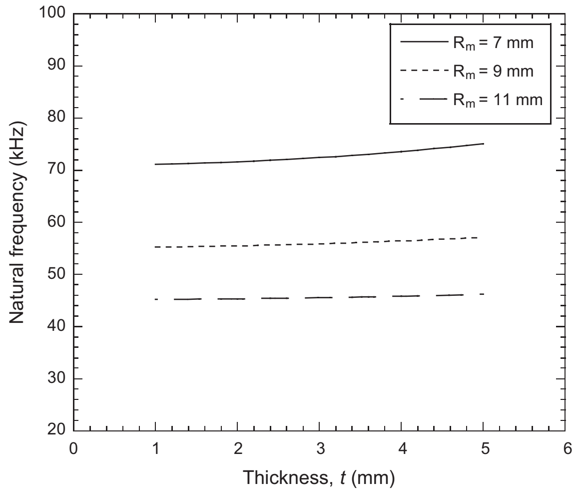

Fig. 6. Piezoelectric natural frequencies of the fundamental mode displayed as a function of the thickness for various mean radii of the cylinder.

5. Conclusion

The vibrational characteristics of piezoelectric cylindrical transducers were studied by deriving a characteristic equation for resonance of radial transducers. The piezoelectric natural frequencies of the transducers were calculated from the theoretical formulae and then compared with experimental values. This comparison verifies that the theoretical results agree well with the experimental results.

Numerical results from theoretical analysis provided information about mode shape, the effect of the piezoelectricity on the natural frequency, and the dependence of the piezoelectric natural frequency on the radius and thickness of the cylinder. As shown, fundamental mode for a cylindrical transducer is similar to a rigid-body mode of a plate. The piezoelectric natural frequency of the fundamental mode was shown to increase as the radius of curvature decreased. The frequency of the fundamental mode did not depend significantly on the thickness of the cylinder.

Acknowledgements

This work was supported by the Soongsil University Research Fund.

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